MATH 2010 Advanced Calculus I
Suggested Solutions for Homework 1

11.1 Q58 Find the center C and the radius α for the sphere

$$3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9.$$

Solution

Rewriting the sphere as

$$x^2 + y^2 + z^2 + \frac{2}{3}y - \frac{2}{3}z = 3$$

Complete the squares on the $x-, y-$ and $z-$terms and write

$$x^2 + (y^2 + \frac{2}{3}y + \frac{1}{9}) + (z^2 - \frac{2}{3}z + \frac{1}{9}) = 3 + \frac{2}{9}$$

which is equivalent to

$$x^2 + (y + \frac{1}{3})^2 + (z - \frac{1}{3})^2 = \frac{29}{9}.$$

It follows that the center is $(0, -\frac{1}{3}, \frac{1}{3})$ and the radius is $\frac{\sqrt{29}}{3}$.

11.2 Q58 Show that a unit vector in the plane can be expressed as $\mathbf{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$, obtained by rotating $\mathbf{i}$ through an angle θ in the counterclockwise direction. Explain why this form gives every unit vector in the plane.

Solution

Let $\mathbf{u}$ be any unit vector in the plane. If $\mathbf{u}$ is positioned so that its initial point is at the origin and terminal point is at (x, y), then $\mathbf{u}$ makes an angle θ with i, measured in the counter-clockwise direction. Since $\mathbf{u}$ is a unit vector, then $|\mathbf{u}| = 1$ and

$$x = |\mathbf{u}|\cos\theta = \cos\theta, \qquad y = |\mathbf{u}|\sin\theta = \sin\theta$$

Thus

$$\mathbf{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}.$$

Since $\mathbf{u}$ was assumed to be any unit vector in the plane, this holds for every unit vector in the plane.

11.3 Q4

$$\mathbf{v} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}, \qquad \mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

Solution

$$\mathbf{v}\cdot\mathbf{u} = 2\times 2 + 10\times 2 - 11\times 1 = 13.$$

$$|\mathbf{v}| = \sqrt{2^2 + (10)^2 + (-11)^2} = 15.$$

$$|\mathbf{u}| = \sqrt{2^2 + 2^2 + 1} = 3.$$

The cosine angle between $\mathbf{v}$ and $\mathbf{u}$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{13}{15\times 3} = \frac{13}{45}.$$

The scalar component of $\mathbf{u}$ in the direction $\mathbf{v}$ is

$$|\mathbf{u}|\cos\theta = 3\times\frac{13}{45} = \frac{13}{15}.$$

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{13}{225}(2\mathbf{i}+10\mathbf{j}-11\mathbf{k}) = \frac{26}{225}\mathbf{i}+\frac{26}{45}\mathbf{j}-\frac{143}{225}\mathbf{k}.$$

11.3 Q14 Find the measures of the angles between the diagonals of the rectangle whose vertices are $A=(1,\,0), B=(0,\,3), C=(3,\,4)$ and $D=(4,\,1)$.

Solution

Note that

$$\vec{AC} = (2,\,4), \qquad \vec{BD} = (4,\,-2)$$

Then

$$\vec{AC}\cdot\vec{BD} = 2\times 4 - 4\times 2 = 0$$

which implies the angles between the diagonals are $90°$.

[11.3 Q30] In real-number multiplication, if $uv_1 = uv_2$ and $u \neq 0$, we can cancel the u and conclude that $v_1 = v_2$. Does the same rule hold for the dot product? That is, if $\mathbf{u}\mathbf{v}_1 = \mathbf{u}\mathbf{v}_1$ and $\mathbf{u}\neq 0$, can you conclude that $\mathbf{v}_1 = \mathbf{v}_2$? Give reasons for your answer.

Solution

No, $\mathbf{v}_1$ need not equal $\mathbf{v}_2$. For example, $\mathbf{i}+\mathbf{j}\neq\mathbf{i}+3\mathbf{j}$. However,

$$\mathbf{i}\cdot(\mathbf{i}+\mathbf{j}) = 1\times 1 + 0\times 1 = 1$$

and

$$\mathbf{i}\cdot(\mathbf{i}+3\mathbf{j}) = 1\times 1 + 0\times 3 = 1.$$

[11.4 Q8]

$$\mathbf{u} = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}, \qquad \mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

Solution

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

The length of $\mathbf{u}\times\mathbf{v}$ is $|\mathbf{u}\times\mathbf{v}| = \sqrt{2^2+2^2+2^2} = 2\sqrt{3}$, and the direction is

$$\frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|} = -\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}.$$

Since

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

Then the length of $\mathbf{v} \times \mathbf{u}$ is $2\sqrt{3}$ and the direction is $\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$.

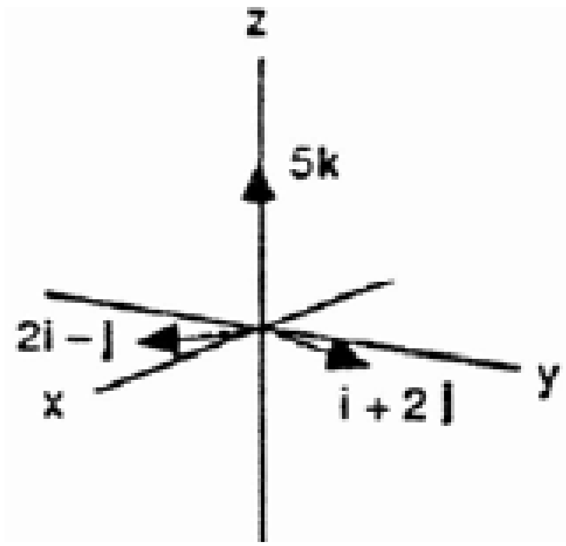
[11.4 Q12]

$$\mathbf{u} = 2\mathbf{i} - \mathbf{j}, \qquad \mathbf{v} = \mathbf{i} + 2\mathbf{j}.$$

Solution

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 5\mathbf{k}.$$

Graph



[11.4 Q16]

$$P(1,\ 1,\ 1), \quad Q(2,\ 1,\ 3), \quad R(3,\ -1,\ 1).$$

We first compute

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}.$$

The area of the triangle determined by the points P, Q and R is

$$\frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{\sqrt{4^2 + 4^2 + 2^2}}{2} = 3.$$

A unit vector perpendicular to plane PQR is given by

$$\frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \frac{1}{6}(4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

[11.4 Q34] If $\mathbf{u} \neq 0$ and if $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

Solution

Yes. If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then

$$\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = 0, \qquad \mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = 0$$

Suppose now that $\mathbf{v} \neq \mathbf{w}$. Then $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = 0$ implies that $\mathbf{v} - \mathbf{w} = k\mathbf{u}$ for some real number $k \neq 0$. This in turn implies that

$$\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot k\mathbf{u} = k|\mathbf{u}|^2 = 0.$$

It follows that $\mathbf{u} = 0$. Since $\mathbf{u} \neq 0$, it cannot be true that $\mathbf{v} \neq \mathbf{w}$, so $\mathbf{v} = \mathbf{w}$.

[11.4 Q38]

$$A(-6,\, 0), \quad B(1,\, -4), \quad C(3,\, 1), \quad D(-4,\, 5)$$

Solution

Since

$$\vec{AB} = (7,\, -4), \qquad \vec{AD} = (2,\, 5)$$

Then

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -4 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 43\mathbf{k}.$$

The areas of the parallelograms determined by A, B, C and D is

$$|\vec{AB} \times \vec{AD}| = 43.$$